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MULTIDIMENSIONAL POLYNOMIAL INTERPOLATION

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Abstract. The article is devoted to the practical application of multidimensional polynomial interpolation. Various areas of application of solutions to the interpolation problem are discussed, such as astronomy, solid state mechanics, aerodynamics, meteorology, crystallography, optics, computer graphics, medical tomography, geographic information systems, photography, and others. Different methods based on the using of the interpolation Lagrange polynomial are considered, such as linear interpolation, cubic interpolation, bilinear interpolation and bicubic interpolation. Algorithms have been developed and a program code has been written for an approximate solution to the problem of image scaling based on bilinear and bicubic interpolation. A numerical experiment on image transformation is carried out. On the base of the detailed comparative analysis of the obtained results, the conclusions are made about the capabilities of the methods under consideration.

Key words: interpolation; Lagrange polynomial; bilinear interpolation; bicubic interpolation; image conversion.

INTRODUCTION

Unknown intermediate values foundation of a function is an important practical problem. It is often necessary to operate with data sets obtained experimentally or by random sampling in scientific and engineering calculations. The method of recovery of a function from an existing discrete set of its known values in order to approximate its unknown intermediate values is known as interpolation of function. This task has been described by many authors in various sources [1–5]. With the advent and development of computer technology, it has been necessary to use and modify numerical methods to solve the interpolation problem. Specialists in various professions often need to perform a large number of calculations with the smallest error. There are many ways to solve the interpolation problem [6–8]. Each method has its own advantages, disadvantages and specifics. As a rule, the main criteria for the quality of methods are the error, the amount of input data required for the given accuracy, the computational complexity and the execution time of the algorithm.

But interpolation problem is incorrect one. The problem of computing results reliability is under consideration of different authors [9–14].

FORMULATION AND SOLUTION OF THE INTERPOLATION PROBLEM

Description of the methods and formulation

If some function $f(x)$ is given by its values $y_j=f(x_j)$ on a discrete set of points x_j, $j=0,...,m$, and it is necessary to approximately determine the analytical form of this function in order to be able to compute this function at intermediate points $x\in(x_j, x_{j+1})$, an algebraic polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i$ is usually used as the interpolating function. And since the polynomial $P_n(x)$ at the nodal points must coincide with the given values of the function $y_j=f(x_j)$, the problem is reduced to the solution of the system of linear algebraic equations

$$\sum_{i=0}^{n} a_i x_j^i = y_j, j = k, \dots, k+n \quad (1)$$

relatively the unknowns a_i, where k is the number of the initial nodal point used in the calculation. The solution of the system (1) is, in particular, the well-known formula of the interpolation Lagrange polynomial

$$P_n(x) = L_n(x) = \sum_{j=k}^{k+n} y_j \prod_{\substack{j=k \\ i \neq j}}^{k+n} \frac{x - x_i}{x_j - x_i}. \tag{2}$$

According to the formula ((2) the Lagrange interpolation polynomial of the first degree is written as the following formula:

$$L_1(x) = \frac{x - x_{k+1}}{x_k - x_{k+1}} y_k + \frac{x - x_k}{x_{k+1} - x_k} y_{k+1}.$$

A graphical interpretation of the linear interpolation is presented in figure 1.

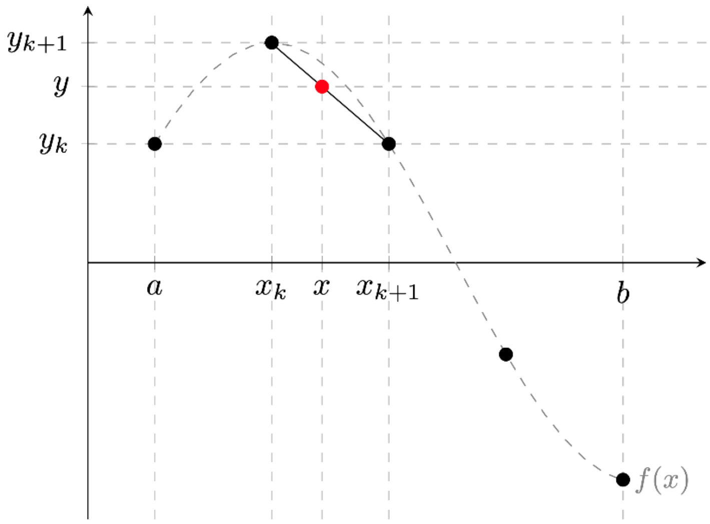

Fig. 1 Linear interpolation.

The Lagrange polynomial of the third degree obtained by formula ((2) can be represented as the following expression:

$$L_3(x) = \frac{(x - x_{k+1})(x - x_{k+2})(x - x_{k+3})}{(x_k - x_{k+1})(x_k - x_{k+2})(x_k - x_{k+3})} y_k +$$
$$+ \frac{(x - x_k)(x - x_{k+2})(x - x_{k+3})}{(x_{k+1} - x_k)(x_{k+1} - x_{k+2})(x_{k+1} - x_{k+3})} y_{k+1} +$$
$$+ \frac{(x - x_k)(x - x_{k+1})(x - x_{k+3})}{(x_{k+2} - x_k)(x_{k+2} - x_{k+1})(x_{k+2} - x_{k+3})} y_{k+2} +$$
$$+ \frac{(x - x_k)(x - x_{k+1})(x - x_{k+2})}{(x_{k+3} - x_k)(x_{k+3} - x_{k+1})(x_{k+3} - x_{k+2})} y_{k+3}.$$

It is easy to notice that the structure of these formulas is such that for each nodal point $x=x_j$ of the nodal points included in the set used by the formula, only one term isn't equal to zero, and it is the one that includes y_j. Therefore, $L_n(x_j) = y_j$.

BILINEAR INTERPOLATION

Applications

Bilinear interpolation is a generalization of linear interpolation to three-dimensional space. The fields of application of the bilinear interpolation method are quite diverse. The method is used in computer graphics and physical disciplines. In computer graphics the method is used to increase or decrease the resolution of images. In embedded control systems, the method is often applied to extract values from data sets. In natural sciences it is used in the processing of numerical data.

Description of the method

The essence of this method is the following.

1) The desired point is calculated from a grid of four points:

$$Q_{1,1} = (x_1,y_1),\ Q_{1,2} = (x_1,y_2),\ Q_{2,1} = (x_2,y_1),\ Q_{2,2} = (x_2,y_2).$$

2) Linear interpolation from surrounding points is applied to find the auxiliary points R_1, R_2:

$$f(R_1) \approx \frac{x_2-x}{x_2-x_1} f(Q_{1,1}) + \frac{x-x_1}{x_2-x_1} f(Q_{2,1}),$$

$$f(R_2) \approx \frac{x_2-x}{x_2-x_1} f(Q_{1,2}) + \frac{x-x_1}{x_2-x_1} f(Q_{2,2}).$$

3) Interpolation is used to find the desired point by the auxiliary points:

$$f(P) \approx \frac{y_2-y}{y_2-y_1} f(R_1) + \frac{y-y_1}{y_2-y_1} f(R_2).$$

A graphical interpretation of the bilinear interpolation is presented in figure 2.

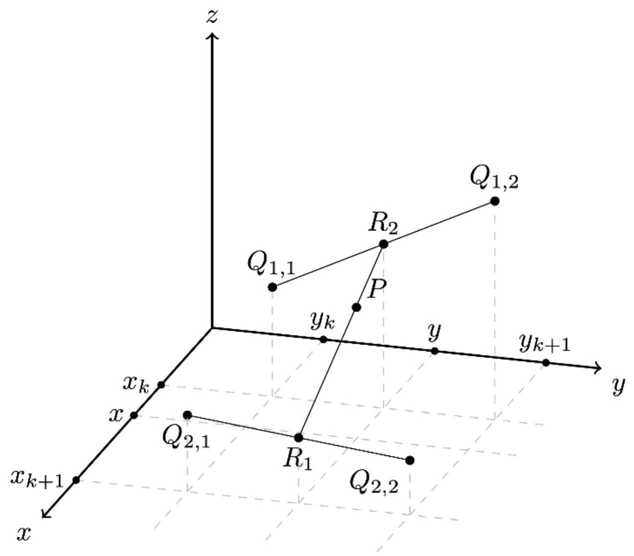

Fig. 2 Bilinear interpolation.

BICUBIC INTERPOLATION

Applications

Bicubic interpolation is a generalization of cubic interpolation. It is used to solve an extensive list of tasks. For example, bicubic interpolation is used in solid mechanics to approximate its surface basing on geometric parameters. This makes it possible to determine stresses and deformations at specific points on the surface.

Bicubic interpolation is also often used in aerodynamics for approximate calculations of air flow parameters on the surface of an object. With the help of this method, drag coefficients and other parameters that affect the aerodynamic properties of object are determined.

In crystallography bicubic interpolation is applied to calculate crystal structures basing on their geometric parameters, which is necessary to determine the properties of crystalline materials, such as thermal conductivity and optical properties.

Bicubic interpolation is a good tool for approximate computing of the parameters of lenses in optics, which allows determining of the properties of optical systems, such as focal lengths and aberrations.

Bicubic interpolation is used in medical tomography to reconstruct a three-dimensional model of organs and tissues based on images. This makes it possible to determine the size and shape of organs, as well as to detect pathologies and other abnormalities.

In geoinformatics bicubic interpolation is applied to predict the properties of the earth's surface basing on its geometric parameters. In this way, elevation maps are created, deposits are predicted, and other geoinformation studies are carried out.

Bicubic interpolation is used in astronomy to calculate the parameters of astronomical objects to determine distances, sizes, and other quantities.

In meteorology bicubic interpolation helps to approximate the parameters of atmospheric modeling. This allows determining the temperature, pressure, and other parameters of the atmosphere at specific points.

Bicubic interpolation is also used in photography to enlarge or decrease the size of an image. This makes it possible to preserve the quality of the image as it is resized, filling in the missing pixels based on neighboring pixels.

Description of the method

The generalization of cubic interpolation to three-dimensional space is carried out in the following way.

1) The desired point is calculated on a grid of 16 points Q_{ij} for i, j = 1, 2, 3, 4.
2) Cubic interpolation is applied to find points R_i for i=1, 2, 3, 4.
3) Cubic interpolation is used to define the desired point P from the auxiliary points.

A graphical interpretation of bicubic interpolation is presented in figure 3.

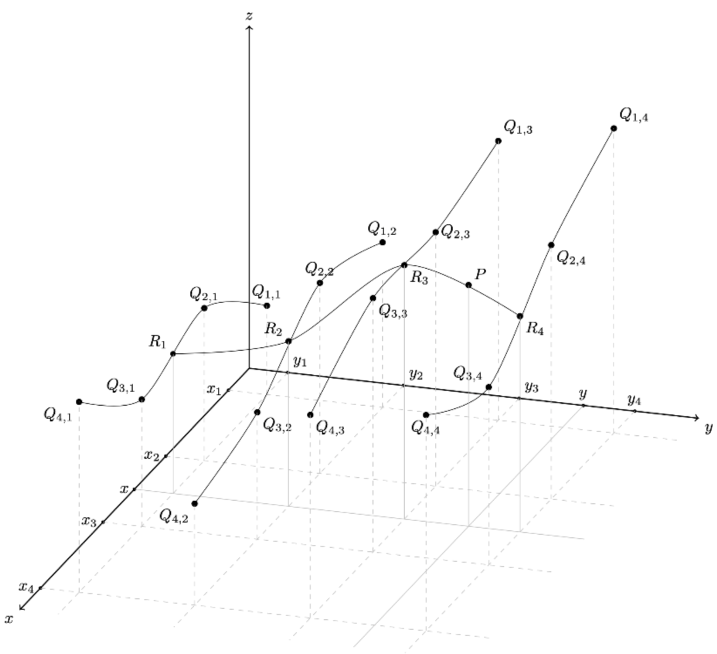

Fig. 3 Bicubic interpolation.

IMAGE SCALING

There are many tools for image processing, one of the simplest one is scaling by using interpolation.

Image interpolation occurs at some stage in all digital raster photographs. It is necessary for resizing, unfolding an image from one grid of pixels to another, etc.

Different interpolation methods can be applied to different subsets of the set of initial points to achieve optimal results for solving of the problem. All algorithms for solving the interpolation problem can be divided into two categories: adaptive and nonadaptive.

The advantageous combination of methods is called the adaptive interpolation. It is worth noting, that among the nonadaptive methods, the bilinear interpolation is one of the simplest and the fastest one. The adaptive methods change depending on the subject of the interpolation, whereas the nonadaptive methods treat all pixels in the same way. Thus, an example of using adaptive interpolation in an image scaling problem is an algorithm that scales the dark areas of an image using bilinear interpolation and the light areas using bicubic interpolation. At the same time, dark monochrome images scaled by bilinear interpolation are sharper.

One iteration of calculation of intermediate points with help of bicubic interpolation works with a large number of points. Therefore, bicubic interpolation produces smoother images than bilinear interpolation and is optimal in terms of processing time and output quality. For this reason, it is one of the standard methods used when writing image editing software, printer drivers, and camera drivers.

Fig. 4 Original image.

a *b*

Fig. 5 A doubling in size by:
a – bilinear interpolation; *b* – bicubic interpolation.

In order to demonstrate the operation of the described methods of bilinear and bicubic interpolation, let's take, for example, the following Figure 4. The initial data for the operation of the software written for the developed algorithms corresponding to the described scaling methods is an image in .png format. Let's consider how it is further transformed and compare the results. The doubling of size of the original image by bilinear interpolation, then by bicubic interpolation, is shown in Figure 5.

A comparison of the images obtained by the two considered methods shows that after bilinear interpolation, the image is more grainy and "pixelated" than after cubic interpolation. This becomes more noticeable at greater scaling. After applying the bicubic interpolation method, the picture is smoother, but "soapy". We can conclude that this is due to the number of points used for the next iteration. Some details are lost, but for scaling it is more qualitative.

Conclusions

Thus, several methods of multidimensional interpolation are described. The application of these methods is in demand in a wide range of tasks. When it comes to interpolating functions of multiple variables, generalizations of polynomial interpolation to *n*-dimensional spaces retain the idea of interpolating points in one plane, then another, and so on. Illustrative results obtained with the help of the developed algorithm for enlarging images using the described methods are shown. Depending on the images and their sizes, one method may be better, and worse than the other, and adaptive interpolation may give better results.

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МЕТАДАННЫЕ / METADATA

Название: Многомерная полиномиальная интерполяция.

Аннотация: Статья посвящена практическому применению многомерной полиномиальной интерполяции. Обсуждаются различные сферы применения решений задачи интерполяции, такие как астрономия, механика твёрдого тела, аэродинамика, метеорология, кристаллография, оптика, компьютерная графика, медицинская томография, геоинформационные системы, фотография и другие. Рассматриваются различные методы, основанные на применении интерполяционного многочлена Лагранжа, такие как линейная, кубическая, билинейная и бикубическая интерполяция. Составлены алгоритмы и написан программный код для приближенного решения задачи масштабирования изображений на основе билинейной и бикубической интерполяции. Проведен численный эксперимент по преобразованию изображений. На основе приведенного подробного сравнительного анализа полученных результатов сделаны выводы о возможностях рассматриваемых методов.

Ключевые слова: интерполяция; полином Лагранжа; билинейная интерполяция; бикубическая интерполяция; преобразование изображений.

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