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Comparison of accuracy of the Cauchy problem solutions by different numerical methods

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Abstract. The solution of the Cauchy problem for an ordinary first-order differential equation by different numerical methods is under consideration. As an example, a differential equation with a known exact solution is chosen. This test sample examines the variation of solution error of the Cauchy problem. The Euler method, the improved Euler method, the predictor-corrector method and the Runge–Kutta method of the fourth accuracy order are investigated. The comparative analysis of the error estimates of the calculations results obtained by different methods is carried out. Then, a numerical filtering method is used for all computed results in order to refine them. The calculated values are compared with the filtered ones, and the error is estimated.
Key words. Cauchy problem, Euler method, improved Euler method, predictor-corrector method, Runge–Kutta method, numerical experiment, numerical filtering, error estimate.

Introduction

Foundation a solution to an ordinary first-order differential equation that satisfies a given initial value is known as the Cauchy's problem [1].

It is an integral part of mathematical modeling of various processes and objects. In this regard, many methods have been developed to solve this problem [2–5]. Depending on the power of the computer equipment and the allocated time, one or another solution method is chosen. In the case of a test sample, the criterion for the reliability of the obtained approximate solution is the permissible deviation from the known exact solution. n the absence of an exact solution to the problem, the criterion for the quality of the calculation results is the error estimates obtained on the basis of extrapolation of the results [6–9] or their numerical filtering [10–15]. The Euler method, the improved Euler method, the predictor-corrector method and the Runge–Kutta method of the fourth accuracy order are the most well-known methods for solving the Cauchy problem.

Formulation and description of methods

Euler method

For the numerical solution of the problem, a grid is introduced x_j, $j=0, ..., n$, $x_0 = a$. The value $x_n = b$ is determined by practical need. In the case of uniform partitioning with step $h = (b – a)/n$ we have $x_j = a + jh$.

Then, for the approximate solution of the first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with an initial value $y(a) = y_0$ the method of left rectangles of numerical integration of functions is used

$$y_{j+1} \approx y_j + \int_{x_j}^{x_{j+1}} f[x_j, y(x_j)]\,dx \approx y_j + f(x_j, y_j)h_{j+1},\ \ h_j = x_j - x_{j-1}. \tag{2}$$

Integration is carried out at a constant value $f(x, y)$. According to (1) the value of the function $f(x,y)$ is equal to the derivative of the desired function $y(x)$. It follows that there is a shift along the tangent drawn to the graph of the function $y(x)$ from the starting point. Then, the angle of inclination of the tangent at the resulting point is determined and a new step is performed. This method has the first order of accuracy.

Improved Euler Method

The improved Euler method is based on the idea of the method of middle rectangles of numerical integration of functions. The integral function is fixed at the value corresponding to the middle of the private segment of the integration $[x_j, x_{j+1}]$ and instead of (2) we have:

$$y_{j+1} \approx y_j + \int_{x_j}^{x_{j+1}} f\left[x_{j+1/2}, y(x_{j+1/2})\right] dx \approx y_j + h_{j+1} f\left[x_{j+1.2}, y_{j+1/2}\right],$$

where $x_{j+1/2} = \frac{x_j + x_{j+1}}{2} = x_j + \frac{h_{j+1}}{2}$.

At the same time, the value of the $y(x_{j+1/2})$ is approximately calculated using the usual Euler method:

$$y^*_{j+1/2} \approx y_j + \frac{h_{j+1}}{2} f\left[x_j, y_j\right],$$
$$y_{j+1} \approx y_j + h_{j+1} f\left[x_{j+1/2}, y^*_{j+1/2}\right].$$

When sheared along a straight line drawn from the starting point parallel to the tangent to the mid-point, the resulting error has significantly less value than in the Euler method. The Euler improved method is globally a second-order accuracy method.

Predictor-corrector

The method is based on the trapezoid method of numerical integration of functions

$$y_{j+1} \approx y_j + \frac{h_{j+1}}{2}\left(f\left[x_j, y_j\right] + f\left[x_{j+1}, y_{j+1}\right]\right).$$

Approximate calculation of $f\left[x_{j+1}, y_{j+1}\right]$ is also carried out by the Euler method, i. e. we again have the method that requires the calculation of the function $f(x, y)$ with two argument values (x, y)

$$y^*_{j+1} = y_j + h_{j+1} f\left[x_j, y_j\right],$$
$$y_{j+1} \approx y_j + \frac{h_{j+1}}{2}\left(f\left[x_j, y_j\right] + f\left[x_{j+1}, y^*_{j+1}\right]\right).$$

Thus, the value of y^*_{j+1} can be seen as predicted one, and the value of y_{j+1} as corrected value of the desired function. The method has a second order of accuracy.

Runge–Kutta method of the fourth accuracy order

This method requires more calculations than all the methods described above, but it allows you to achieve greater accuracy in fewer steps:

$$y_{j+1} \approx y_j + \frac{h_{j+1}}{6}(K_1 + 2K_2 + 2K_3 + K_4),$$

where

$$K_1 = f\left[x_j, y_j\right], \qquad K_2 = f\left[x_j + \frac{h_{j+1}}{2}, y_j + \frac{h_{j+1}}{2} K_1\right],$$
$$K_3 = f\left[x_j + \frac{h_{j+1}}{2}, y_j + \frac{h_{j+1}}{2} K_2\right], \qquad K_4 = f\left[x_j + h_{j+1}, y_j + h_{j+1} K_3\right].$$

COMPUTATIONAL EXPERIMENT

Numerical solution of the Cauchy problem by various methods

Let's study the variation of the error of the Cauchy problem solution using a test case, which is differential equation

$$\frac{dy}{dx} = y \sin x \frac{dy}{dx} \tag{3}$$

in the segment $x \in \left[0, \frac{\pi}{2}\right]$ under the initial value $y\ (0) = y_0 = 1$. This problem has a precise solution $y = e^{1-\cos x}$.

The algorithms have been developed and calculations have been carried out for doubling number of partial segments for solution of the problem (3) by all considered methods.

The results of the numerical experiment are given in Table 1, where the difference between the exact and calculated values, i.e., the absolute error, is represented for different *n* for each of the methods at the endpoint of the segment $x = \pi/2$ at which the error has the maximum value.

Table 1

Variation of the absolute error of different methods for variation of n

n	Euler method	Improved Euler method	Predictor-corrector method	Runge–Kutta method
2	–1.1629E+00	–2.1197E-01	–3.0542E-01	–4.1392E-03
4	–7.1538E-01	–6.5206E-02	–8.1597E-02	–2.7604E-04
8	–4.0739E-01	–1.8231E-02	–2.0610E-02	–1.7234E-05
16	–2.1943E-01	–4.8207E-03	–5.1396E-03	–1.0619E-06
32	–1.1420E-01	–1.2391E-03	–1.2803E-03	–6.5616E-08
64	–5.8299E-02	–3.1407E-04	–3.1931E-04	–4.0729E-09
128	–2.9461E-02	–7.9058E-05	–7.9718E-05	–2.5360E-10
256	–1.4809E-02	–1.9832E-05	–1.9915E-05	–1.5818E-11
512	–7.4247E-03	–4.9665E-06	–4.9769E-06	–9.8721E-13
1024	–3.7173E-03	–1.2427E-06	–1.2440E-06	–6.1284E-14
2048	–1.8599E-03	–3.1080E-07	–3.1097E-07	–1.5543E-14
4096	–9.3028E-04	–7.7717E-08	–7.7738E-08	–1.6875E-14
8192	–4.6522E-04	–1.9431E-08	–1.9434E-08	2.6201E-14
16384	–2.3263E-04	–4.8581E-09	–4.8584E-09	–1.3767E-14
32768	–1.1632E-04	–1.2145E-09	–1.2146E-09	–3.5083E-14
65536	–5.8161E-05	–3.0374E-10	–3.0364E-10	2.5313E-14
131072	–2.9081E-05	–7.5978E-11	–7.5862E-11	1.8652E-14
262144	–1.4540E-05	–1.8955E-11	–1.8928E-11	3.2419E-14
524288	–7.2702E-06	–4.7056E-12	–4.7407E-12	2.7534E-14
1048576	–3.6351E-06	–1.2452E-12	–1.2523E-12	–8.3045E-14

A detailed analysis of the numerical results obtained shows that the Euler method, which has the lowest order of accuracy, is significantly inferior in accuracy to any of the considered methods. The results of the improved Euler method and the predictor-corrector are comparable. This is consistent with the fact that they have the same order of accuracy. The best result is shown by the Runge–Kutta method of the fourth accuracy orders. Already for $n = 1024$, the error of the method has the minimum order (10^{-14}) for double-precision calculations. Further accuracy enlargement is not possible due to limited digit capacity.

Numerical filtering of the results of the Cauchy problem solving

Numerical filtering makes it possible to significantly increase of the calculation's accuracy [16], which in comparison with extrapolation methods, does not use assumptions about the smallness of the additional error.

Table 2 shows the error values of the calculation results obtained after the fourth numerical filtering.

Table 2

Results errors of methods after numerical filtering

n	Euler method	Improved Euler method	Predictor-corrector method	Runge–Kutta method
32	-1.8097E-03	-1.0979E-05	-4.4264E-06	-4.1454E-10
64	-2.3483E-04	-1.2279E-07	-7.2938E-07	-2.3050E-11
128	-2.7792E-05	1.0315E-08	-2.1867E-07	1.3731E-12
256	-3.4237E-06	1.2652E-09	1.8296E-08	9.3259E-15
512	-4.2719E-07	1.0023E-10	5.5842E-10	0.0000E+00
1024	-5.3432E-08	6.9531E-12	2.8237E-11	4.4409E-16
2048	-6.6839E-09	4.7073E-13	1.6094E-12	-8.8818E-16
4096	-8.3567E-10	3.9080E-14	5.4623E-14	-1.8208E-14
8192	-1.0461E-10	4.4409E-15	8.3933E-14	-1.5987E-14
16384	-1.3081E-11	-4.0412E-14	-2.6645E-15	-1.6875E-14
32768	-1.7244E-12	5.0626E-14	7.1054E-15	-1.0658E-14
65536	-4.5297E-14	-1.3221E-12	6.2172E-15	-3.4639E-14
131072	3.8636E-14	-8.3933E-14	-7.5495E-15	1.1395E-12
262144	-3.3440E-13	4.2810E-13	5.7288E-14	2.3537E-14
524288	-1.1413E-13	5.5511E-14	1.1724E-13	1.0214E-14
1048576	1.0352E-12	1.0703E-13	1.0796E-12	4.1744E-14

Table 2 shows that for the Euler method the most accurate solution is fixed for $n = 131072$. Note, that numerical filtering allows you to achieve the best result from Table **Ошибка! Источник ссылки не найден.**This is true for all of the considered methods. The results of the improved Euler method and the predictor-corrector method are comparable due to their similar order of accuracy. The methods reach a peak accuracy for $n = 8192$ and $n = 16384$ respectively. Although the results in Table 2 are comparable at each step for the improved Euler method and the predictor-corrector method, the predictor-corrector method loses accuracy much later than the improved Euler method **Ошибка! Источник ссылки не найден.**($n = 131072$ and $n = 8192$ respectively).

The best result is shown by the Runge–Kutta method of fourth orders of accuracy for the smallest number of steps. When we use numerical filtering, this method reached the peak value of accuracy for $n = 512$.

However, all methods are characterized by a decrease in accuracy after the peak value has been reached. Thus, the accumulation of round off error due to the limited digit capacity of mantissa is affected. It is explained by the loss of the lowest digits during the alignment of the orders of the terms, which is necessary for the operation of summation of terms of different order.

The results of the numerical experiment show that in numerical filtering, the accumulation of rounding error occurs according to a statistical law. In this regard, the dependence of the error on n for the Euler method in ordinary summation is well described by the straight line $y = 16.5 – ½\lg n$ (Figure 1, *a*).

In order to eliminate this accumulating error, it can be proposed to replace the summation procedure by accumulation with a pairwise summation procedure, i.e., the summation of pairs of terms of the same order. In pairwise summation, there is no accumulation of rounding off error (Figure 1, *b*).

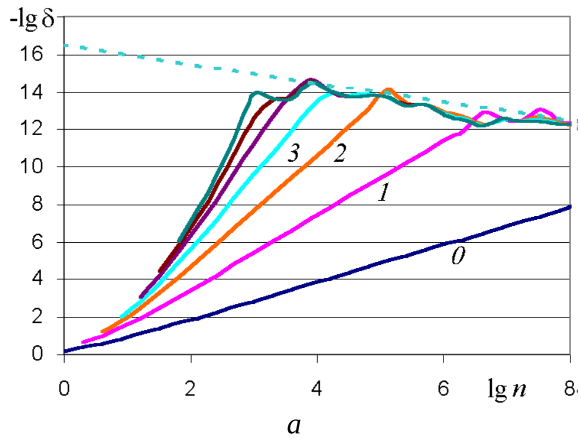

Fig. 1 Filtering the results of the numerical solution of the problem:
a – summation by accumulation; *b* – pairwise summation.

Conclusions

Thus, based on the described methods, the algorithms have been developed for calculating of the approximate solution of the differential equation. Its estimates of the error based on comparison with the exact solution are obtained. The analysis of the results of the numerical solution of the Cauchy problem for different methods presented in the paper shows that the given estimates of the error of the results correspond to the declared orders of accuracy of these methods.

In particular, it can be noted that the Euler method achieves more precise results much slower than the Runge–Kutta method and does it in a greater number of steps. Numerical filtering not only refines the values, but also allows you to achieve better results in fewer steps.

And you can eliminate the accumulation of rounding off error with the help of pairwise summation, which allows you to find optimal solutions considering the number of partial segments *n*, the number of filtering and the high accuracy of the result.

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МЕТАДАННЫЕ / METADATA

Название: Сравнение точности решений задачи Коши различными численными методами.

Аннотация: В статье рассматривается решение задачи Коши для обыкновенных дифференциальных уравнений первого порядка различными численными методами. В качестве примера выбрано дифференциальное уравнение, имеющее точное решение. На данном тестовом примере исследуется изменение погрешности решения для четырех различных методов решения задачи Коши. Это метод Эйлера, усовершенствованный метод Эйлера, предиктор-корректор, метод Рунге–Кутты четвертого порядка точности. Проводится сравнительный анализ погрешностей результатов вычислений, полученных разными методами. Далее для всех результатов применяется метод численной фильтрации с целью их уточнения. Вычисленные значения сравниваются с отфильтрованными, проводится оценка погрешности.

Ключевые слова: задача Коши; метод Эйлера; усовершенствованный; предиктор-корректор; метод Рунге-Кутты; вычислительный эксперимент; численная фильтрация; оценка погрешности.

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