ISSN 2658-5014 (Print)
2024. Vol. 6, No. 2 (17) pp. 33–38
SIIT
SYSTEMS ENGINEERING AND INFORMATION TECHNOLOGIES
Scientific message
ISSN 2686-7044 (Online)
http://siit.ugatu.su

UDC 621.9.047 DOI 10.54708/2658-5014-SIIT-2024-no2-p33 EDN LLRWGJ

NUMBER SYSTEMS
REPRESENTED BY QUADRATIC POLYNOMIALS

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Abstract. The article discusses number systems with irrational bases represented by their minimal quad-ratic polynomial. Using direct encoding methods, it is possible to decompose integers into finite representations with integer coefficients. The algorithm for encoding any integers accord-ing to the irrational number system (the number system with the base of Pisot numbers) is considered, as well as a set of all quadratic polynomials, the leading roots of which are the basis of the number system with finite decomposition is presented. Such number systems give finite decompositions of integers into a floating-point system. The developed algorithm is not inferior in speed to alternative integer decomposition algorithms.

Keywords: Pisot numbers; number systems with irrational bases; direct encoding; polynomial representation of number system base.

INTRODUCTION

Various number systems have long been known and widely used in various technical fields. Knowing that any rational number can be encoded in almost any number system with an integer base, and using well-known conversion principles, it is also possible to try to encode numbers with a fractional or even irrational base of the number system. Such decompositions will be called "floating point decompositions", where the encoded number has an integer and a fractional part. In most cases, decompositions based on such bases will almost always be infinite, but there are special irrational numbers, using which as bases of the number system will result in finite encoding of integers or periodic decomposition if the number is rational. These numbers were first introduced by Axel Thue in 1852 and further described in more detail by G. H. Hardy in 1919 in the context of Diophantine approximation. They gained wide recognition after the publication by Charles Pisot in 1938 under the name of Pisot numbers or PV (Pisot–Vijayaraghavan) numbers. One of the characteristics of these numbers is that their higher powers are close to integers.

THE GOLDEN RATIO AND ITS PECULIARITY

In mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities

$$\frac{a+b}{a} = \frac{a}{b} = \varphi,$$

where φ represents the golden ratio. This irrational number is a solution to the quadratic equation $x^2 - x - 1 = 0$ with roots

$$x_1, x_2 = \frac{1 \pm \sqrt{5}}{2} = \varphi.$$

By the definition of algebraic numbers, the Golden Ratio is an algebraic integer. This illustrates the unique property of the Golden Ratio among positive numbers

$$\frac{1}{\varphi} = 1 - \varphi, \qquad \varphi = 1.618 \ldots, \qquad \varphi^2 = \varphi + 1.$$

The conjugate and defining quadratic polynomials lead to decimal values that have their fractional part in common with φ. In a more general sense, any power of φ, is equal to the sum of the two immediately preceding powers of itself φ

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} = \varphi F_n + F_{n-1}.$$

As a result, it is easy to represent the *n*-th power φ as a polynomial of the first degree. The multiple number and the constant are adjacent Fibonacci numbers. This, in turn, leads to another property of positive powers of φ. If $\left[\frac{n}{2} - 1\right] = m$, then

$$\varphi^n = \varphi^{n-1} + \varphi^{n-3} + \cdots + \varphi^{n-1-2m} + \varphi^{n-2-2m}.$$

When the golden ratio is used as the base of the number system, every integer has a finite representation despite φ being irrational. Any number in the $Q(\varphi)$ field has a periodic representation.

The golden ratio is a fundamental unit of the field of algebraic numbers $Q(\sqrt{5})$and is also the Pisot–Vijayaraghavan number in the same field of numbers.

DECOMPOSITION BY THE GOLDEN RATIO

Let's describe a simple algorithm for converting an integer from the decimal number system to the so-called Fierich number system. First of all, let's discuss that any number can be represented as a number from the $Q(\varphi)$ field in the following way:

$$\boldsymbol{N = a + b\varphi}\ , \tag{1}$$

where, N is any natural number, a and b are integers, φ is the highest root of the quadratic equation $x^2 - x - 1 = 0$. Let's introduce some properties of operations on numbers. In the future, the algorithm for encoding into an irrational number system with a base that is the root of its quadratic polynomial will be based on these properties.

1. $(a + b\varphi) + (c + d\varphi) = (a + c) + (b + d)\varphi.$
2. $(a + b\varphi) - (c + d\varphi) = (a - c) + (b - d)\varphi.$
3. $(a + b\varphi) * (c + d\varphi) = (ac - bd) + (ad + bc + bd)\varphi.$

The next property is not so obvious but can be easily derived.

Let $(a + b\varphi) > (c + d\varphi)$, where $\varphi = \frac{-B + \sqrt{B^2 - 4C}}{2}$ is the highest root of its quadratic polynomial $x^2 + Bx + C = 0$, then

$$(a - \mathrm{c}) > (d - b)\varphi.$$

We substitute φ

$$(a - \mathrm{c}) > (d - b)\left(\frac{-B + \sqrt{B^2 - 4C}}{2}\right)$$

and transform it to remain our irrationality on the right

$$2 * (a - \mathrm{c}) + B(d - b) > (d - b)\sqrt{B^2 - 4C}.$$

Then we get rid of irrationality and move all elements to the left side

$$\boldsymbol{\left(2 * (a - \mathrm{c}) + B(d - b)\right)^2 - (d - b)^2(B^2 - 4C) > 0}. \tag{2}$$

The obtained inequality allows comparing numbers of the form (**1**) among themselves using only integer coefficients, without resorting to irrationalities, which will significantly increase the accuracy of calculations. If the left side of inequality (2) is less than zero, it means that the number $a + b\varphi$ is less than the number $c + d\varphi$.

Using these properties, we can multiply, add, subtract, and compare numbers in the field of our coefficient φ. Note that all actions are performed with integer values.

Let's move directly to the algorithm:

1. Convert the integer number x to a number with base φ, where $x = x + 0 \cdot \varphi$.
2. Calculate the highest power φ close to our x. At the same time, this power is still less than the number x being encoded.
3. To obtain a new number, write 1 in the digit corresponding to the power φ.
4. Subtract our number x from the highest power φ using the second property.
5. If our resulting number is not equal to 0, go back to step 2.
6. Finish the algorithm and get the sequence.

For example, we need to decompose the number 5 from the decimal number system into the Fierichal one:

7. Initially, we have the sequence …0000000.000000….
8. The highest power, not exceeding our number, is $\varphi^3 = 1 + 2\varphi$.
9. Subtract: $5 - (1 + 2\varphi) = 4 - 2\varphi$, our sequence becomes 1000.0000…
10. The highest power φ not exceeding our resulting value$\varphi^{-1} = \varphi - 1$.
11. Subtract it from $4 - 2\varphi . 4 - 2\varphi - (-1 + \varphi) = 5 - 3\varphi$, put a one in the necessary digit 1000.100000…
12. Find the power for our number $\varphi^{-4} = 5 - 3\varphi$, subtract it from our number, and get 0 in the result.
13. Substitute a one in the – 4 digit and get the sequence $1000.1001_{\varphi} = 5_{10}$.

In general, there are many irrational numbers like the golden ratio that lead to a finite decomposition. As mentioned above, such numbers are called the Pisot–Vijayaraghavan numbers or PV numbers [3].

PISOT NUMBERS

The set of Pisot numbers consists of real algebraic integers $P > 1$ (hereinafter P is Pisot number) such that all algebraic conjugates of these numbers lie inside the unit circle. In other words, a Pisot number is the principal root of its minimal polynomial of degree n in the form:

$$x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \dots + A_0 = 0,$$

where $A_{n-1}, A_{n-2}, \dots, A_0$ are integers.

In this article, we will show several quadratic irrationalities, where the principal roots of these minimal polynomials will be Pisot numbers (Table).

In fact, it is possible to find all Pisot numbers for which the minimal polynomial is quadratic. Let's consider the general form of a quadratic polynomial, where the principal root will be a Pisot number, and its conjugate will be located within the unit circle (by definition).

Table

Quadratic irrationalities and the roots

Roots	Polynomial	Numerical value
$\frac{1+\sqrt{5}}{2}$	$x^2 - x - 1 = 0$	1.618033…
$1+\sqrt{2}$	$x^2 - 2x - 1 = 0$	2.414214…
$\frac{3+\sqrt{5}}{2}$	$x^2 - 3x - 1 = 0$	2.618033…
$1+\sqrt{3}$	$x^2 - 2x - 2 = 0$	2.732050…
$\frac{3+\sqrt{13}}{2}$	$x^2 - 3x - 1 = 0$	3.302775…

Let's consider a quadratic polynomial $ax^2 + bx + c = 0$, where the coefficients $a = 1$, b и c are integers. We impose the following conditions on the roots. The principal root $x_1 > 1$, and the conjugate root $|x_2| < 1$. This results in the system of inequalities

$$\begin{cases} \{b^2 - 4c > 0, \\ \frac{-b+\sqrt{b^2-4c}}{2} > 1\,, \\ \left|\frac{-b-\sqrt{b^2-4c}}{2}\right| < 1. \end{cases} \quad (3)$$

Solving this system, we obtain a region of points on the axes relative to b is the x-axis and c is the y-axis. In Figure below, we see the regions constructed from the solutions of the system (3).

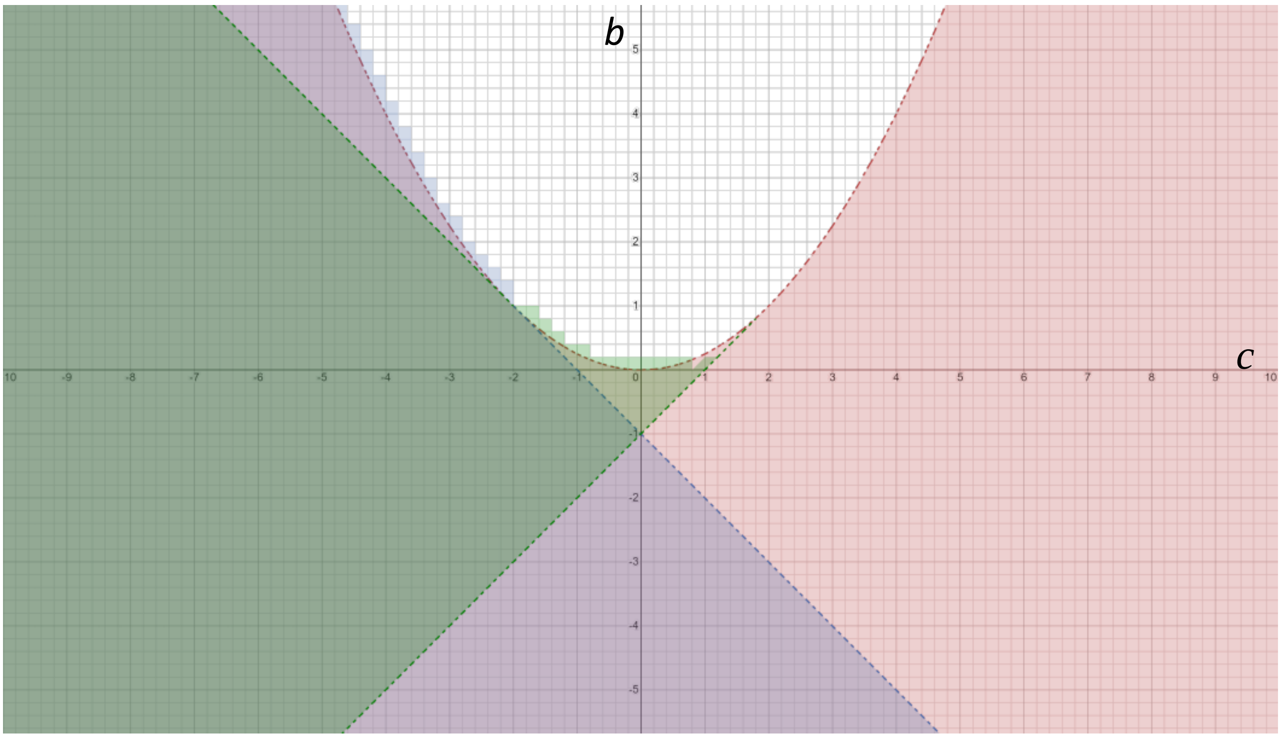

Fig. Solution of the system (3).

Conclusions

Thus, the solution obtained from the system allows us to find the region of all possible coefficients of polynomials that form Pisot–Vijayaraghavan numbers. These numbers will start with the golden ratio.

Note that the root of the golden ratio 1.618… is the only minimal PV number in the set of quadratic polynomials [1]. The proof of this is quite obvious, it only requires a slight change in the conditions we imposed earlier on the system (3).

By using such PV numbers, we can convert any numbers from the decimal number system to the so-called Fierich floating-point system.

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Поступила в редакцию 20 декабря 2023 г.

МЕТАДАННЫЕ / METADATA

Название: Системы счисления, представимые квадратичным полиномом.

Аннотация: Рассматриваются системы счисления с иррациональным основанием, представимые своим минимальным квадратичным полиномом. Используя методы прямого кодирования, можно раскладывать целые числа в конечные представления с целыми коэффициентами. Рассмотрен алгоритм кодирования любых целых чисел по иррациональной системе счисления (система счисления с основанием чисел Пизо), а также представлено множество всех квадратичных полиномов, старшие корни которых являются основанием системы счисления с конечными разложениями. Такие системы счисления дают конечные разложения целых чисел в систему счисления с плавающей точкой. Разработанный алгоритм не уступает по скорости работы с альтернативными алгоритмами разложения по целочисленным системам счисления.

Ключевые слова: числа Пизо; системы счисления с иррациональным основанием; прямое кодирование; полиномиальное представление основания системы счисления.

Язык статьи / Language: английский / English.

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