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VERIFICATION OF THE NUMERICAL FILTERING METHOD RESULTS BY CALCULATION WITH INCREASED BIT WIDTH

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Abstract. The problem of reliability and error estimation in computational results when solving mathematical modelling problems by numerical methods is under consideration. Using the example of results for the Stokes soliton problem obtained by different authors, it is shown that the reliability of published results and error estimation can only be substantiated with more accurate results. An approach is proposed that allows for an independent verification and confirmation of the correctness of the estimates obtained with help of numerical filtering method. This method is based on an alternative numerical method that provides enhanced accuracy due to an increase in the computational bit width.

Keywords: error estimation, computational experiment, numerical filtering, numerical-analytical method, Stokes soliton.

INTRODUCTION

In most modern mathematical modeling problems, solutions can be obtained only using numerical methods and by means of complex computations with specialized software systems. Therefore, the accuracy of published results can be evaluated only when more precise results become available. However, the chronology of obtaining more precise results can be inconsistent, and thus the issue of justifying obtained error estimates remains relevant even with this method of verification.

This is clearly seen in the example of solving the problem of a solitary wave or Stokes soliton. The Stokes soliton problem was not accidentally chosen as a "testing ground" for analysis. During the 20th century, many scientists tried to refine and improve the solution to this problem by proposing various numerical methods and improved mathematical models. However, the task turned out to be difficult to solve, and research in this area led to different, often contradictory, results.

Several key parameters characterize the Stokes soliton: amplitude, mass, impulse, potential and kinetic energy, circulation, and the Froude number. In shipbuilding, the Froude number Fr is of great importance and is used for comparing wave formation conditions for vessels of different sizes.

The Stokes soliton has been extensively studied in many works [1–6]. Results were published by M. S. Longuet-Higgins and J. D. Fenton in 1974 [7], yielding a Froude number of $Fr = 1286$. In 1977, M. J. H. Fox computed $Fr = 1286$ [8]. J. M. Williams in 1981 presented a result of $Fr = 1290889$ [9], using linear extrapolation of parameter dependencies to refine the results. J. K. Hunter and J.-M. Vanden-Broek in 1983 published $Fr = 1290906$ [10], then W. A. B. Evans and M. J. Ford in 1996 obtained $Fr = 129089053$ [11]. D. V. Maklakov in 2002 reported $Fr = 12908904558$ [12]. In the works of V. P. Zhitnikov and N. M. Sherykhalina [13–16], several results were shown, the latest two $Fr = 12908904558634$ and $Fr = 129089045586335$ were obtained using numerical filtering.

The application of the filtering method has already been described in detail [17–22]. Briefly, the method is based on a priori knowledge of the error behavior as a function of discretization parameters. The error is represented as a sum of several terms of known form but with unknown coefficients, and numerical filtering involves the sequential (or group) suppression of these error components.

Unlike the acceleration of sequence convergence and extrapolation (the idea of which is based on the fulfillment of certain conditions), numerical filtering is only aimed at providing additional information for subsequent comparative analysis. Sequential (or group) suppression of the error components makes it possible to obtain several additional sequences to the original sequence, the comparison of which makes it possible to make estimates of the error, as well as to conclude that they are valid. It should be noted that the approach under consideration is purely heuristic and does not have a strict mathematical proof.

THE PROBLEM STATEMENT

A solution of the type of solitary wave at which its amplitude has the maximum value is considered. The fluid is assumed to be ideal. The Stokes wave corresponds to an internal breaking angle which is equal to $2\pi/3$. The force of gravity directed vertically downward. The velocity of the fluid flow at infinity is V_∞ and the thickness of the jet at infinitely distant points *A* and *C* is *h* (fig. 1, *a*). At points *A*, *B*, and *C*, the pressure *P* is equal to the atmospheric pressure P_0.

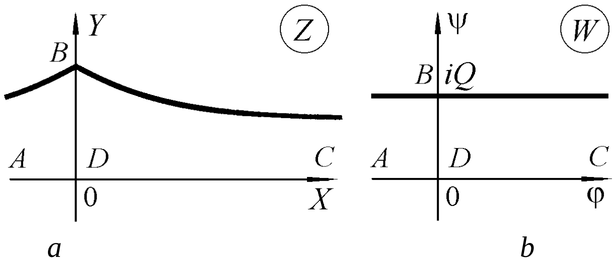

Fig. 1 Shapes of the region corresponding to the flow in planes:
a – physical plane; b – plane of the complex potential

Initially, the shape of the free surface of the flow is unknown. The Bernoulli equation for $P=P_0$ is used to relate the ordinate of a point *Y* and the value of the modulus of the velocity vector *V* at the free boundary *ABC*. Denoting the gravitational acceleration by *g*, the Bernoulli equation can be written as follows

$$\left(\frac{V}{V_\infty}\right)^2 + \frac{2Y}{Fr^2 h} = \text{const}, \; Fr = \frac{V_\infty}{\sqrt{gh}}. \tag{1}$$

Figure 1, *b* presents the complex potential plane *W*. In this plane, the flow region is a strip.

We introduce an auxiliary plane of the parametric variable to solve the problem (fig. 2, *b*). The region corresponding to the flow in the physical plane is mapped onto this plane, which is then connected by a conformal mapping to the complex potential plane. We use the standard notation: $Z=X=iY$, *X* and *Y* are coordinates in the Cartesian system, and *W* is a function of the complex potential.

Let's write the function $W(\chi)$ as follows

$$W(\chi) = \varphi + i\psi = Qw(\chi) = -Q\chi + Q,$$

where $Q = hV_\infty$ is the flow rate of the fluid.

We non-dimensionalize $z = Z/h$. We denote modulus of the velocity vector by *V* and the angle at which the velocity vector is inclined to the axis *OX* by θ.

Let us consider the Zhukovsky function

$$\omega = i\ln\frac{dw}{dz} = \theta + i\tau,$$

which represents the logarithmic hodograph of the velocity.

Thus, the complex conjugate of the non-dimensional velocity is

$$\frac{dw}{dz} = \frac{V}{V_\infty}\mathrm{e}^{-i\theta}, \;\; \tau = \ln\frac{V}{V_\infty}.$$

The conditions for the function $\omega(\chi)$ are the follows

a) the Bernoulli equation (1) aligns the real and imaginary parts of $\omega(\chi)$ for $\mathrm{Im}\,\chi = 0$;

b) the real part Re $\omega(\chi)=\theta = 0$ for Re $\chi = 0$, $0 < \text{Im}\,\chi \leqslant 1$; (2)
c) the real part Re $\omega(\chi) = 0$ for Im $\chi = 1$; (3)
d) the real part Re $\omega(\chi) \to \pm\,\pi/6$ for $\chi = \sigma + i0$, $\sigma \to \pm 0$;
e) $\omega(\chi) \to 0$ as $\chi \to \infty$.

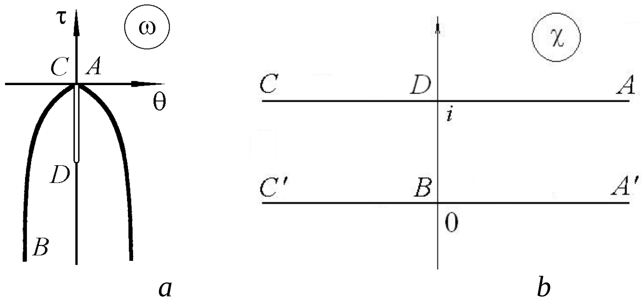

Fig. 2 Boundaries of the flow regions in planes:
a – logarithmic hodograph of velocity; b – parametric variable plane χ

NUMERICAL-ANALYTICAL METHOD

Let the function

$$\omega(\chi) = \omega_1(\chi) + \omega_2(\chi),$$

where $\omega_2(\chi)$ is a function that satisfies conditions b), c), d), and e).

The first term is reconstructed using the Schwarz formula

$$\omega_1(\chi) = -i \int_0^\infty \text{Re}\omega_1(\sigma) \frac{\text{sh}\,\pi\sigma}{\text{ch}\,\pi\sigma - \text{ch}\,\pi\chi} d\sigma.$$

For $\chi \to \sigma_m + i0$ using the Sokhotski formula, we get

$$\omega_1(\sigma_m) = -i \cdot v.p. \int_0^\infty \text{Re}\omega_1(\sigma) \frac{\text{sh}\,\pi\sigma}{\text{ch}\,\pi\sigma - \text{ch}\,\pi\sigma_m} d\sigma + \text{Re}\omega_1(\sigma_m),$$

where *v.p.* is the integral principal value.

For $0 \leqslant m < n$ the principal value of the integral is computed using the formula

$$v.p. \int_0^{\sigma_n} \text{Re}\omega_1(\sigma) \frac{\text{sh}\,\pi\sigma}{\text{ch}\,\pi\sigma - \text{ch}\,\pi\sigma_m} d\sigma =$$

$$= \int_0^{\sigma_n} \text{Re}\omega_1(\sigma) \frac{\text{Re}\omega_1(\sigma)\text{sh}\,\pi\sigma - \text{Re}\omega_1(\sigma_m)\text{sh}\,\pi\sigma_m}{\text{ch}\,\pi\sigma - \text{ch}\,\pi\sigma_m} d\sigma -$$

$$-\text{Re}\omega_1(\sigma_m)\sigma_m + \frac{1}{\pi}\text{Re}\omega_1(\sigma_m)\ln\frac{e^{\pi\sigma_n} - e^{\pi\sigma_m}}{e^{\pi\sigma_n} - e^{-\pi\sigma_m}}.$$

Numerical integration is performed using the Gaussian quadrature method.

The term $\omega_2(\zeta)$ accounts the singularity in the solution at the point $\chi = 0$ (fig. 2). The Bernoulli equation (1) can be written as

$$e^{3\tau}\frac{d\tau}{d\sigma} - \frac{1}{Fr^2}\sin\theta = 0. \qquad (4)$$

The function $\omega_2(\chi)$ is expressed as

$$\omega_2(\chi) = \frac{i}{3}\ln f(\chi) + iC_1\left[\left(f(\chi)\right)^{\beta} - 1\right] =$$

$$= \frac{i}{3}\ln\frac{1 - e^{-\pi\chi}}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^2} + iC_1\left(\frac{1 - e^{-\pi\chi}}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^2}\right)^{\beta} - iC_1. \qquad (5)$$

The function (5) satisfies condition (2) at *AD*, condition (3) at *DB*, and equation (4) for $\chi \to 0$. Therefore, we have

$$\frac{d}{d\chi}\omega_2(\chi) = -\frac{\pi}{6}\frac{1}{\operatorname{ch}\frac{\pi\chi}{2}} + i\frac{\pi}{3}\frac{1}{\operatorname{sh}\pi\chi} + i\beta C_1\left(\frac{1 - e^{-\pi\chi}}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^2}\right)^{\beta-1}\pi e^{-\frac{\pi\chi}{2}}\frac{e^{-\frac{\pi\chi}{2}} + i}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^3}.$$

For $\chi = \sigma + i0$, $\sigma \to 0$ we can write

$$\omega_2(\sigma + i0) = \frac{i}{3}\ln\frac{1 - e^{-\pi\sigma}}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^2} + iC_1\left(\frac{1 - e^{-\pi\sigma}}{\left(1 + ie^{-\frac{\pi\chi}{2}}\right)^2}\right)^{\beta} - iC_1 = \theta_2(\sigma) + i\tau_2(\sigma),$$

$$\frac{d\theta_2}{d\sigma} = C_1\frac{\pi}{2}\beta\sin\frac{\pi\beta}{2}\left(\frac{\pi\sigma}{2}\right)^{\beta-1} - \frac{\pi}{6} + O\left(\sigma^{\beta}\right),$$

$$\frac{d\tau_2}{d\sigma} = \frac{1}{3\sigma} + C_1\frac{\pi}{2}\beta\cos\frac{\pi\beta}{2}\left(\frac{\pi\sigma}{2}\right)^{\beta-1} + O\left(\sigma^{\beta}\right).$$

Substituting these expressions into equation (4), and equating terms of the same order, we have the following equations

$$C_1 - \tau_1(0) = -\frac{1}{3}\ln\left(\frac{3}{\pi}\frac{1}{Fr^2}\right), \qquad (6)$$

$$(\beta + 1)ctg\frac{\pi\beta}{2} = \frac{1}{\sqrt{3}}. \qquad (7)$$

We solve the equation (7) and determine the value of $\beta \approx 0{,}80267907$.

The following expression is integrated

$$\frac{dZ}{h} = dz = dx + idy = e^{i\omega(\chi)}d\chi.$$

This is necessary for determining the shape of the free boundary. In this problem, the midpoint rule of numerical integration is applied, and the results are filtered, that increases the accuracy of the solution up to an order of 10^{-15}.

NUMERICAL SOLUTION

The problem is solved numerically using the collocation method. It is required that the Bernoulli's equation (1) be satisfied at discrete set of points along the axis BA' ($\sigma_m = Fm/n$, $m = 0, \ldots, n-1$). Additionally, the equation (6) must be satisfied. As a result, we obtain a system of $n + 1$ nonlinear equations with respect to parameters Fr, C_1, θ_m ($m = 0, \ldots, n-1$). To approximate the solution of this system, a modification of the Newton method is used, where the sum of squared residuals is minimized in all equations to regulate the step size. When the modulo of the residuals became less than 10^{-30}, the computing process is stopped. A mantissa length corresponding to thirty-five decimal places approximately is used to represent the numbers.

NUMERICAL RESULTS AND RESULTS PROCESSING

The following parameters of the Stokes soliton are computed and filtrated (the table 1).

Table 1

Computations and filtering results

Parameters of soliton	Results required verification (filtered)	Results of the offered method application (computed)	Results of the offered method application (filtered)
Froude number	1.2908904558634	1.2908904558635	1.290890455863341
amplidude	0.8331990845196	0.8331990845197	0.833199084519532
mass	1.9703206601317 (−2)	1.97032066009	1.97032066013185
impuls	2.5434681351545	2.54346813510	2.54346813515454
circulation	1.7145692405337 (2)	1.7145692405	1.71456924053350
kinetic energy	0.5350088359709	0.53500883596	0.53500883597099
potential energy	0.4376726934439	0.437672693441	0.43767269344390

The estimates are obtained using the Schwarz integral with an integral boundary condition (the Bernoulli equation). This allows us to refine the value of the Froude number Fr up to $2\text{-}3\cdot10^{-15}$. The filtration results of the described solution are shown in fig. 4, where points of the group 0 represent the error in the computed results, and points of groups 1 and 2 represent the errors of the first and second filtration, respectively.

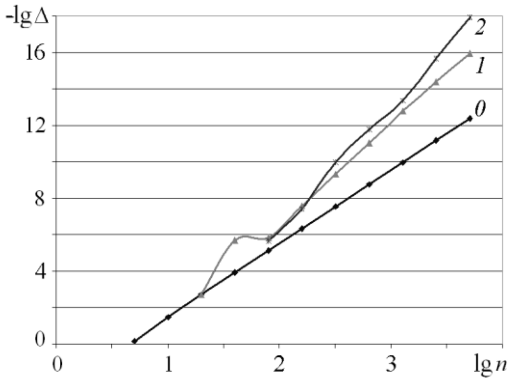

Fig. 3 Filtering results for the Froude number

CONCLUSIONS

The use of the new method and the extended mantissa allowed us to reduce the error to 10^{-16} (fig. 4). A comparison with the value obtained earlier shows a difference about $-1{,}5\times10^{-15}$. The set of described actions has allowed us to find a new, more accurate value for the Froude number, equal to $Fr = 1{,}2908904558633395 \pm 10^{-16}$.

Thus, the obtained solution confirms the previously calculated estimates. Moreover, the developed program, which is used to solve the problem with this numerical method, shows a significant improvement in computational speed.

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МЕТАДАННЫЕ / METADATA

Название: Верификация результатов, полученных методом численной фильтрации на вычислениях с увеличенной разрядностью.

Аннотация: В работе рассматривается проблема достоверности и оценки погрешности результатов вычислений при решении задач математического моделирования численными методами. На примере результатов задачи о солитоне Стокса, полученных разными авторами, показано, что обосновать достоверность и оценить погрешность опубликованного результата можно только более точным результатом. Предложен подход, который позволяет провести независимую проверку и подтвердить корректность оценок, полученных в рамках метода численной фильтрации, основанный на применении альтернативного численного метода, обеспечивающего повышенную точность за счет увеличения разрядности вычислений.

Ключевые слова: оценка погрешности, вычислительный эксперимент, численная фильтрация, численно-аналитический метод, солитон Стокса

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